EQUATIONS AND VARIABLES ASSOCIATED WITH THE LINEAR DIFFERENTIAL EQUATION.

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SECOND ARTICLE.—GEOMETRICAL INTERPRETATIONS.

Any one wishing to read on the geometry of space higher than three dimensions is referred to an article by G. Veronese published in the Mathematische Annalen, Bd. XIX, 1882, entitled " Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens," also to a book on the subject by the same author. Another helpful contribution in Italian is " Studio sulle quadriche in uno spazio linare ad un numero quelnuque di dimensione," by Corrado Segre (Memoire della Realla Accademia Delle Scienze di Torino, Tomo. XXXVI, 1885).

A few definitions and theorems relative to such a space will be introduced here :

Consider a space of $n - 1$ dimensions S_{n-1}, the homogeneous coordinates of an element of which are $x_1, x_2, x_3, \ldots, x_n$.

In a space of $n - 1$ dimensions, k relations between the n homogeneous coordinates represent a surface in $n - 1 - k$ dimensions S_{n-1-k}. If $k = n - 2$, the coordinates are functions of a single parameter and we have a curve S_1. If $k = n - 1$ we have a limited number of points S_0.

Every element of a linear space S'_{n-1} can be determined if we take in place of the $x_1, x_2, \ldots, x_n$, as coordinates $y_1, y_2, \ldots, y_n$, where the y's are proportional to given functions of the x's, which are linear, homogeneous, and independent ; that is

$$\rho y_i = \sum_1^n l_{ik} x_k \,.$$

Whatever be the nature of the elements of this space they are called points.

A plane is a surface in which all the k relations are linear. A plane S_1' is a straight line, and S_0' a unique point. Instead of considering the surfaces as the locus of points, we can consider them as the envelope of planes S'_{n-2}. We classify them, then, according to the number of parameters on which the tangent planes S'_{n-2} depend. When S'_{n-2} depends on a single parameter it envelopes a developable $\triangle$.

When $m \leqq n$, every space of $m - 1$ dimensions S_{m-1} will be contained in S_{n-1}, and among those notably the linear spaces S'_{m-1}. S'_{m-1} will be linear in the sense intended here when it consists of all those points whose coordinates x_i considered in the ultimate space S_{n-1} are given linear homogeneous functions of the my's ; that is

$$x_i = \sum_1^m l_{ik} y_k \,.$$

From this it follows that in order that the x be coordinates of the points of a space linear in $m - 1$ dimensions, it is necessary and sufficient that they satisfy certain $n - m$ linear homogeneous equations.

In general, a space S'_{n-1-l} is composed of those points whose coordinates x satisfy l linear homogeneous equations, and hence the ∞^{l-1} equations formed by combining these l equations linearly, or of points whose coordinates can be represented as linear homogeneous functions of $n - l$ other quantities (variable coordinates of the points in S'_{n-1-l}). S'_{n-1-l} is then the intersection of $\infty^{l-1} S'_{n-2}$.

A space S'_{n-2} is particularized by its equation

$$\xi_1 x_1 + \xi_2 x_2 + \ldots - \xi_n x_n = 0$$

in which the ξ are n given coefficients. Given $n - 1$ such equations, they serve to determine the ratio of the ξ.

Thus the S_{n-2} of points in S'_{n-1} can in their turn serve as the elements of a space Σ, also linear and of $n - 1$ dimensions. The ξ will be the homogeneous coordinates of a S'_{n-2} considered as elements of Σ.

An arbitrary linear homogeneous equation between the coordinates ξ of a S'_{n-2} necessitates that this pass through a fixed point, of which the coordinates are the coefficients of the ξ. Let

$$x_1 \xi_1 + x_2 \xi_2 + \ldots + x_n \xi_n = 0 \tag{1}$$

be this equation ; it may be called the equation of the point x_i in plane coordinates ξ_i.

From (1) we see that when the ξ's are constant we have a plane S'_{n-2} in S_2 which is an element of Σ ; and when the x's are constant we have a Σ'_{n-2} in Σ which is an element of S. Through this duality a proposition proven for one yields another for the other by an interchange of planes S'_{n-2} and points S'_0. Σ'_m will denote a linear space of m dimensions in Σ. Its elements are planes S'_{n-2} with the ξ for coordinates.

Every S'_{n-k} is the intersection common to a Σ'_{k-2}. Two such linear spaces are called conjugate. Two linear spaces S'_{n-k} and S'_{k-2}, the x alone being variable, are called dual.

If we have given $k+1$ points $x_i', x_i'', x_i''', \ldots, x_i^{(k+1)}$ $(i = 1, 2, \ldots, n)$ of a S_k' it is determined and any other of its points has the coordinates

$$x_i = h'x_i' + h''x_i'' + h'''x_i''' + \ldots + h^{(k+1)}x_i^{(k+1)},$$

where the $h^{(\nu)}$ can be considered as the $k+1$ homogeneous coordinates of a point in a space of k dimensions S_k'. Here the accents affixed to h and x serve merely to distinguish different quantities and have no necessary connection with differentiation. Similarly, $k+1$ elements S'_{n-2} with coordinates $u_i', u_i'', \ldots, u_i^{(k+1)}$ $(i = 1, 2, \ldots, n)$ determine a Σ_k'. The coordinates of any element S'_{n-2} in Σ_k' can be represented by

$$u_i = g'u_i' + g''u_i'' + \ldots + g^{(k+1)}\, u_i^{(k+1)}.$$

The g's may serve as the coordinates of an arbitrary plane S'_{n-2} in Σ_k'.

As we can take all the planes S'_{n-2} of S as elements of a space different from S, so whatever k may be, if $< n-1$, we may take all the S_k' as elements of another space. If S_k' is determined by the points $x_i, x_i'', \ldots, x_i^{(k+1)}$ Clebsch has shown* that we can take as coordinates of S_k' the determinants of order $k+1$ formed from the matrix

$$\left|\begin{matrix} x_1' & x_2' & \ldots & x_n' \\ x_1'' & x_2'' & \ldots & x_n'' \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{(k+1)} & x_2^{(k+1)} & \ldots & x_n^{(k+1)} \end{matrix}\right|,$$

and analogously, we can determine the coordinates of a Σ' from the $k+1$ S'_{n-2} elements which determine Σ_k'.

The coordinates $u_1, u_2, \ldots, u_n$ of a plane S'_{n-2} are, therefore, the n first minors of the determinant Δ of the nth order,

$$\Delta = \left|\begin{matrix} x_1' & x_2' & \ldots & x_n' \\ x_1'' & x_2'' & \ldots & x_n'' \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{(n)} & x_2^{(n)} & \ldots & x_n^{(n)} \end{matrix}\right| \equiv (x_1'x_2'' \ldots x_n^{(n)}).$$

When we consider a space Σ'_{n-2-k} conjugate to S_k' it will be determined

* Ueber die Fundamentalaufgabe der Invarianttheorie. (Abhandlungen der Kön. Gesellschaft der Wissenchafter zu Göttingen, Band XVII, 1872.)

by $n - 1 - k$ planes S'_{n-2} with coordinates $u_i', u_i'', \ldots, u_i^{(n-1-k)}$ $(i = 1, 2, \ldots, n)$. The coordinates of Σ'_{n-2-k} will be the minors of the matrix

$$\begin{vmatrix} u_1' & u_2' & \ldots & u_n' \\ u_1'' & u_2'' & \ldots & u_n'' \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n-1-k)} & u_2^{(n-1-k)} & \ldots & u_n^{(n-1-k)} \end{vmatrix}$$

and these by a well known theorem in determinants are proportional to the complementary minors of Δ, that is,

$$(u_1'u_2'' \ldots u_{n-1-k}^{(n-1-k)}) = \Delta^{n-2-k}(x_{n-k}^{(n-k)} \ldots x_n^{(n)}) .$$

Thus the coordinates of two conjugates spaces are proportional.

Considering determinants similar to

$$\begin{vmatrix} x_1' & x_2' & \ldots & x_n' \\ x_1^{(k+1)} & x_2^{(k+1)} & \ldots & x_n^{(k+1)} \\ x_1' & x_2' & \ldots & x_n' \\ x_1^{(k+1)} & x_2^{(k+1)} & \ldots & x_n^{(k+1)} \end{vmatrix}, \qquad k < n - 2.$$

it is easy to see that these coordinates are connected by relations and are, therefore, not all independent.

Two planes S_m' S_r' are determined respectively by $m + 1$ and $r + 1$ points. These points when $m + r \leqq n - 2$ will determine a S'_{m+r+1} containing S_m' and S_r'. S_m' and S_r' have, in general, no point common. When they have a S_a', that is, $a + 1$ points common, there remain only $m + r + 1 - a$ distinct points which determine a S'_{m+r-a}. Vice versa, if through S_m' S_r' we can pass a S'_{m+r-a}, then for arbitrary m and r S_m' and S_r' will intersect in a S_a'.

When $S'_{m+r-a} \equiv S'_{n-1}$ then $m + r - n + 1 = a$. If $a = 0$, S_m' and S_r' have one point common; if $a < 0$, they have no points common. If $a = n$, then S_m' contains S_r'; when $m > r$, $a \leqq r$; and when $n + r < n - 1$ and S_m', S_r' have points common, then they are contained in a space of dimensions less than $n - 1$.

When a point moves according to some algebraic law in only two directions it describes a curve S_1. This is of the mth order if it is cut by a S_{n-2} in m points. A curve of the mth order S_1^m can lie only in an S_2, S_3, or S_m; for, were it contained in a S_{m+1}, and not in a lower space S_m, then we could cut it

by a S_m, which can meet it in only m points, and through these m points and another point of the curve pass a S_m. This S_m will cut the curve in $m + 1$ points, which is impossible.

A space S_m is called an algebraic space of order p when every linear space S'_{n-1-m} has with it, in general, p common points. It is indicated by S_m^p. Similarly, Σ_m^p will denote an algebraic space of class p, when any Σ'_{n-1-m} has in common with this Σ_m^p p planes S'_{n-2}, that is, that pass through an S'_{m-1}. An S_{n-2} is a surface, an S'_{n-2} is a surface of the first order. A Σ_{n-2} is an envelope, a Σ'_{n-2} is an envelope of the first class. An equation of degree g in x coordinates will represent a surface of order g, while the same in u coordinates will represent a surface of class g. r equations of degrees $p_1, p_2, \ldots, p_r$ in x coordinates determine a space S^k_{n-1-r}, in which the order k equals the product $p_1, p_2, \ldots, p_r$. If the coordinates were u we should have a $\Sigma^{p_1, p_2, \ldots, p_r}_{n-1-r}$ of class $p_1, p_2, \ldots, p_r$.

The tangents of a curve generate a two-dimensional developable surface; the osculating planes are tangent to this surface, and the osculating spaces of higher orders can be called osculating spaces of this two-dimensional surface.

The osculating planes S_2' describe a three-dimensional developable surface D_3. The osculating spaces S_3 of the curve are tangential spaces of its three-dimensional developable, etc.

Two algebraic spaces S_m^r $S_{m'}^p$ do not in general intersect when $m + m' < n - 1$. When $m + m' \overline{\gtrless} n - 1$ they intersect in a $S^{rp}_{m+m'-n+1}$, and so on for a greater number of spaces.

An S_m^g has only in common with S_k' an $S^g_{m+k+1-n}$. If S_k' has still a point outside $S^g_{m+k+1-n}$, it will intersect S_m^g in a space of at least one dimension more, that is, an $S_{m+k+2-n}$. When $n > i > m$, if S_m^g be contained in S_i, then every S'_{n-1-m} will intersect S_m^g in g points contained in S'_{i-m} in which it intersects S_i'. Vice versa, if every S'_{n-1-m} intersects S_m^g in g points of an S'_{i-m}, that is, in an S_0^g of S'_{i-m}, then every S'_{n-m} intersects S_m^g in an S_1^g of an S'_{i-m+1}, and every S'_{n-m+1} intersects S_m^g in an S_2^g of an S'_{i-m+2}; finally every S'_{n-1} intersects S_m^g in an S_m^g of an S_i', that is S_m^g is contained in S_i'. Then the conditions necessary and sufficient in order that S_m^g may be contained in S_i' is that every S'_{n-1-m} intersect S_m^g in g points of an S'_{i-m}. g points are always contained in an S'_{g-1}, and the condition will always be satisfied when $i - m \overline{\gtrless} g - 1$ or $i \overline{\gtrless} m + g - 1$. Then it follows, that every S_m^g is always contained in a linear space of $m + g - 1$ dimensions (but can also lie in a linear space of p dimensions where $m < p < m + g$).

Every S_m^2 is contained in an S'_{m+1}.

Every S_1^g is contained in an S_v', where $v \leqq g$.

Hence the S_1^g can be distinguished in $g - 1$ species, according as they lie

in an S_g', or in an S'_{g-1}, or in an S'_{g-r}, etc., or finally in an S_2'. Thus there are two species of S_1^3 (curves of the third order), those which lie in an S_3' and those which lie in an S_2'.

It will not be difficult to apply this to the particular case in which the n homogeneous coordinates of a point in S_{n-1} are $y_1^{(v)}, y_2^{(v)}, \ldots, y_n^{(v)}$, solutions of a linear differential equation of the nth order. $v = 0, 1, 2, \ldots$, etc., these upper marks denoting differentiation as in my first paper,

$$u_1 y_1 + u_2 y_2 + \ldots + u_n y_n = 0 \tag{2}$$

is the equation of a plane S'_{n-2}. The u being functions of a single variable x, this plane will envelope a developable surface D_{n-1}, to form which we must add the equation

$$u_1' y_1 + u_2' y_2 + \ldots + u_n' y_n = 0\,. \tag{3}$$

When we also add

$$u_1'' y_1 + u_2'' y_2 + \ldots + u_n'' y_n = 0 \tag{4}$$

we obtain an S'_{n-4}, which will envelope a D_{n-3}. The S'_{n-2}'s osculate D_{n-3}, that is, contain three successive S'_{n-4}.

Adding further

$$u_1''' y_1 + u_2''' y_2 + \ldots + u_n''' y_n = 0\,, \tag{5}$$

we have an S'_{n-5} enveloping a D_{n-4}, and so on until we arrive at

$$u_1^{(n-2)} y_1 + u_2^{(n-2)} y_2 + \ldots + u^{(n-2)} y_n = 0 \tag{n}$$

k being equal to any of the numbers $1, 2, 3, \ldots, n-1$, the k equations (2), (3), (4), ..., $(k+1)$ represent an S'_{n-k-1} of which the surface enveloped is a D_{n-k}. The $n-1$ equations represent a point of which the locus is a curve called the edge of regression of all the developables considered.

When we consider the S'_{n-2}'s as elements of a space Σ'_{n-1}, the u's being the coordinates of these elements, we have the $n-1$ equations

$$y_1^{(k)} u_1 + y_2^{(k)} u_2 + \ldots + y_n^{(k)} u_n = 0\,, \quad k = 0, 1, 2, \ldots, n-2.$$

and a series of propositions similar to the above.

When the y's are taken as the homogeneous coordinates of a point in S_{n-1}, as x varies this point describes a curve Γ belonging to S_{n-1} (Halphen's attached curve). Similarly, the point of which the u's are coordinates will describe a curve C.

To the points of the curve C correspond the tangents S'_{n-2} of the developable D_{n-1}, or the S'_{n-k} osculating the curve Γ, and to the points of Γ correspond Σ'_{n-k} osculating C. The S'_{n-k-1} osculating the one curve correspond to the Σ'_{k-1} osculating the other $(k = 1, 2, \ldots, n-1)$.

The curve Γ is not altered by the substitutions

$$z = f(x)\,. \qquad\qquad v = y\,\frac{1}{\varphi(x)}\,.$$

Thus to a single curve belong an infinite number of equations.

For a single value of x, y can have an infinite number of values, all, however, expressible as linear combinations of the primitive values.

If we imagine x extended over a surface (a plane, or a Rieman's surface) on which the coefficients of the differential equation are one-valued, then to every closed path on this surface will correspond a certain linear substitution with constant coefficients

$$y_i' = \sum_1^n{}_k\, a_{ik} y_k\,.$$

To the totality of all the closed paths that can be drawn on this surface will correspond a certain group of linear substitutions of the y_i (monodromic group) which will be discontinuous. Thus Γ is determined only so far as it is one of a system derived from each other by homographic substitutions.

Thus we have a collineation of S'_{n-1}. Every branch of the functions y_i is changed into another branch of the same functions.

It has been shown* that when a linear differential equation is self-adjoint and of odd order, there always exists a homogeneous relation between the fundamental solutions y_i, $\varphi(y) = 0$; that is, the curve Γ is situated on a surface of the second degree represented by $\varphi(y) = 0$.

The equations

$$u_1^{(k)} y_1^{(l)} + u_2^{(k)} y_2^{(l)} + \ldots + u_n^{(k)} y_n^{(l)} = 0$$

$$(k = 0, 1, 2, \ldots, n-2\,;\ l = 0, 1, 2, \ldots, n-2\,;\ k + l \gtrless n - 1.)$$

show that the points y_i', y_i'', y_i''', etc. $(i = 1, 2, \ldots, n)$ lie on the surface $\varphi(y) = 0$, and also on the tangent at a point distinct from the point of contact. The tangent is then entirely situated on the surface.

As y_1'', y_2'', $\ldots$, y_n'' are points on the osculating surface S_2', and $y_1^{(k)}$, $y_2^{(k)}$, $\ldots$, $y_n^{(k)}$ are points on the osculating surface S_k' $(k = 2, 3, 4, \ldots)$, we see that these surfaces osculating the curve Γ lie entirely on $\varphi(y) = 0$. Γ is then an asymptotic line of the surface.

When n is even we consider the complex

$$\Sigma_i\,(\Sigma_k a_{ik} y'_k)\, y_i = 0\,.$$

We see from this that the surface corresponding to the point y_k' in a cor-

* Vol. IX, No. 6, p. 173.

relative transformation with respect to the complex contains the points $y_k^{(l)}$ $\left[\begin{smallmatrix} k=1,2,\ldots,n \\ l=0,1,\ldots,n-2 \end{smallmatrix}\right]$, that is, is an osculating plane of the curve at the point considered. We have then the geometrical problem, To find curves such that their osculating planes in any point are the planes corresponding to this point in the complex.

The y's being homogeneous coordinates of the points on the curve Γ, and the u's the homogeneous coordinates of the points on C, it is seen that these curves correspond dually to one another.

We do not know that the equations corresponding to these two curves are adjoint, but we can say that by multiplying the y's by some factor they become adjoint.

When $A = 0$ is self-adjoint then the curve C must belong to the same system as Γ. This defines self-adjointness geometrically.

The following is given as an illustration: Let $n = 5$; then $\varphi(y) = 0$ becomes

$$2y_1'y_5 + 2y_2y_4 - y_3^2 = 0\,,$$

which may be written

$$y_1y_5 + y_2y_4 - y_3y_3 + y_4y_2 + y_5y_1 = 0\,; \tag{1}$$

then

$$y_1y_5' + y_2y_4' - y_3y_3' + y_4y_2' + y_5y_1' = 0\,, \tag{2}$$

$$y_1y_5'' + y_2y_4'' - y_3y_3'' + y_4y_2'' + y_5y_1'' = 0\,, \tag{3}$$

$$y_1y_5''' + y_2y_4''' - y_3y_3''' + y_4y_2''' + y_5y_1''' = 0\,, \tag{4}$$

$$y_1'y_5' + y_2'y_4' - y_3'y_3' + y_4'y_2' + y_5'y_1' = 0 = \varphi(y')\,,$$

$$y_1'y_5^{\text{iv}} + y_2'y_4^{\text{iv}} - y_3'y_3^{\text{iv}} + y_4'y_2^{\text{iv}} + y_5'y_1^{\text{iv}} = 0\,,$$

$$y_1'y_5'' + y_2'y_4'' - y_3'y_3'' + y_4'y_2'' + y_5'y_1'' = 0\,.$$

From these equations we see

a) The curve Γ lies on $\varphi(y) = 0$;

b) The tangent plane at y contains y', y'', and y''', and therefore osculates the curve.

$\varphi(y) = 0$ has the generating line S_1', viz:

$$y_1 + by_3 - cy_4 = 0\,,$$

$$y_2 - ay_3 + cy_5 = 0\,,$$

$$y_3 - ay_4 + by_5 = .0\,,$$

where a, b, and c are arbitrary functions of x. This generates the surface from the point of view of linear generators, while the plane S_2' generates it as

an enveloping tangent. Substituting the values of y_1, y_2, y_3 in the polar plane of the point u, and equating the coefficients of y_4 and y_5, S_2' is

$$u_1 + bu_3 - cu_4 = 0\,,$$

$$u_2 - au_3 + cu_5 = 0\,.$$

The intersection of S_1' with the next consecutive generator is given by

$$y_3 \frac{db}{dx} = y_4 \frac{dc}{dx}\,, \quad y_3 \frac{da}{dx} = y_5 \frac{dc}{dx}\,.$$

From which we have

$$\frac{y_3}{\frac{dc}{dx}} = \frac{y_4}{\frac{db}{dx}} = \frac{y_5}{\frac{da}{dx}} = t, \text{ say,}$$

where t is arbitrary.

Then it follows that

$$y_1 = cy_4 - by_3 = tc\frac{db}{dx} - tb\frac{dc}{dx}\,,$$

$$y_2 = ay_3 - cy_5 = t\left[a\frac{dc}{dx} - c\frac{da}{dx}\right],$$

$$y_3 = t\frac{dc}{dx}\,,$$

$$y_4 = t\frac{db}{dx}\,,$$

$$y_5 = t\frac{da}{dx}\,.$$

If these values are to satisfy $\varphi(y) = 0$, the exponent of x in c must be made to depend on those of a and b.